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2004 J. Phys. A: Math. Gen. 37 8619

(http://iopscience.iop.org/0305-4470/37/36/004)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 8619–8637 PII: S0305-4470(04)76778-5

Generalized coherent states for spinning relativistic particles

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Received 26 February 2004
Published 24 August 2004
Online at stacks.iop.org/JPhysA/37/8619
doi:10.1088/0305-4470/37/36/004

Abstract
We construct generalized coherent states of the massless and massive representations of the Poincaré group. They are parametrized by points on the classical state space of spinning particles. Their properties are explored, with special emphasis on the geometrical structures on the state space.

PACS numbers: 03.65.Fd, 02.40.Yy

1. Introduction

The Poincaré group defines the basic symmetry of non-gravitational physics. Every physical system on Minkowski spacetime—quantum fields, in particular—carries a representation of the Poincaré group. Any such representation may be written as a direct sum of irreducible representations. In physical terms, an irreducible representation corresponds to an elementary system characterized by group under study.

A full classification of the representations of the Poincaré group was first achieved by Wigner, in his famous 1939 paper [1]. Remarkably (but not unexpectedly), the irreducible representations correspond to spinning particles. Except for unphysical and degenerate cases, the irreducible representations either describe particles with finite mass M and spin equal to $\frac{n}{2}\hbar$, or massless particles of spin $\frac{n}{2}\hbar$ and of either positive or negative helicity. This result implies that any relativistic system, such as a quantum field, may be analysed in terms of constituent particles, a fact making more plausible the field-particle duality that lies at the heart of quantum field theory.

The analysis of a relativistic system into elementary constituents is not an exclusive quantum mechanical feature. It is also present in classical mechanics. Any state space carrying a symplectic Poincaré group action may be decomposed into elementary systems (corresponding to transitive actions of the group) [2]. Similarly to the quantum case, these

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elementary systems correspond to spinning particles. The only difference is that the quantum description forces the particle spin to take discrete values.

The classical state space Γ and the quantum Hilbert space H of a physical system are related by means of the coherent states, namely an overcomplete family of normalized vectors on H, labelled by points of Γ that satisfy a resolution of the unity. The present paper deals with the construction of generalized coherent states corresponding to the spinning relativistic particles. For that purpose we exploit the fact that the action of the unitary operators representing group elements on a reference vector defines a set of generalized coherent states. We make a convenient (Gaussian) choice for the reference vector and show that the representations of the Poincaré group define generalized coherent states for the spinning relativistic particles, in full correspondence with the results of the classical analysis. We then study the properties of those states. A correspondence of classical functions to quantum operators needs the existence of a resolution of the unity. Even though the Poincaré group leads to a fully covariant family of Hilbert space vectors, a resolution of the unity may be defined only by restricting on spatial hypersurfaces Σ. This procedure breaks the full Poincaré covariance. This is the reason that the natural position operators (like the Newton–Wigner one [3]) for relativistic particles do not transform covariantly under the Poincaré group, even though the corresponding classical functions do.

It needs to be emphasized that the massless and massive cases are very different. The state space for massless particles is not simply the limit $M \to 0$ of the massive ones. It is a different symplectic manifold, with different natural parameters for the physical degrees of freedom, which may be conveniently described in terms of naturally complex variables (twistors).

We place particular emphasis on the geometry of the classical state space, which is induced by the generalized coherent states. In particular, we identify a Riemannian metric on the (extended) state space. Its role is twofold. First it determines the resolution of phase-space measurements thus implementing the Heisenberg uncertainty relation [4]. Second, it is a crucial ingredient of the coherent state path integral [5, 6], because it defines a Wiener process through which the path integral may be regularized.

This is not the first time that generalized coherent states of relativistic particles have been constructed in the literature. There exist, however, substantial differences between earlier work and ours. We should emphasize that the generalized coherent states we construct here are obtained from the representation theory of the Poincaré group and the parameter state space is identified with the classical symplectic manifold that described spinning relativistic particles, and may be obtained, for instance, as coadjoint orbits of the Poincaré group [2].

A complete and rigorous mathematical construction of a large class of generalized coherent states of the Poincaré group has been achieved in [7]—see also previous work [8]. Many families of generalized coherent states for massive relativistic particles are constructed in these papers, without a specification of the reference vector. The relevant parameter space, however, is not the classical state space of a spinning relativistic particle $\mathbf{R}^6 \times S^2$, but rather the state space of a spinless relativistic particle $\mathbf{R}^6$, with the spin degrees of freedom being treated as discrete variables. The properties of those generalized coherent states are different from the present ones—the distinction between massless and massive particles is not emphasized.

Another construction of generalized coherent states of massive spinning particles may be found in [9]. This work involves the representation theory of the group $SU(2) \times SU(2)$ and they are therefore very different in structure from the present ones. A construction of relativistic generalized coherent states within the general theory of wavelets may be found in [10], in which the generalized coherent states are labelled by points of a complexified Minkowski spacetime—interpreted as the classical state space. Reference [11] has dealt with the Moyal representation for spinning relativistic particles, on the same state space with our generalized coherent states. Finally, a precursor of our construction for the massive spinless particles may be found in [4].

The plan of the paper is as follows. In section 2 we provide the necessary background. This involves the structure of the Poincaré group, the basics of two-component spinors and some basic facts about coherent states. In section 3 we construct the generalized coherent states for massive particles and in section 4 for massless ones.

2. Background

2.1. The Poincaré group

The Poincaré group is the semi-direct product of the Lorentz group and R^4, the Abelian group of spacetime translations on Minkowski spacetime. An element of the Poincaré group is the pair $\left(\Lambda^{\mu}{}_{\nu}, C^{\mu}\right)$, which acts on points X^{μ} of Minkowski spacetime as follows:

$$X^{\mu} \rightarrow \Lambda^{\mu}{}_{\nu} X^{\nu} + C^{\mu}. \tag{2.1}$$

In classical mechanics the state space is represented by a symplectic manifold. For this reason we seek group actions on that manifold that preserve the symplectic structure . In quantum mechanics the role of the state space is played by a complex Hilbert space. We seek group actions that preserve the linearity structure and the inner product of the Hilbert space, namely unitary group representations.

When the Poincaré group acts on the phase space Γ of a physical system by symplectic transformations, its Lie algebra is represented by functions on Γ through the Poisson bracket. Writing the generators of the Lorentz transformations as $M_{\mu\nu}$ and of the spacetime translations as P^{μ}, we may define the Pauli–Lubanski vector W^{μ} as

$$W^{\mu} = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_{\nu} M_{\rho\sigma}. \tag{2.2}$$

Analogous operator relations hold in the quantum case. The elementary systems—the ones that correspond transitive actions classically and irreducible representations quantum mechanically—are classified by means of two physical quantities, which are invariant under the action of the Poincaré group. The first such invariant is the rest mass $M := \sqrt{P_{\mu}P^{\mu}}$ and the second one is the spin $s := \sqrt{-\frac{1}{M^2} W^{\mu} W_{\mu}}$. In the classical case spin takes any positive value, while in quantum mechanics discrete values $s = \frac{n}{2}\hbar$, for any non-negative integer n.

2.2. Spinors

In this section we will provide some basic expressions for the spinor calculus, which are necessary in our treatment.

The motivation for spinors comes from the realization that one may define a self-adjoint complex 2×2 matrix $x_{A'A}$ for each 4-vector X^{μ} on Minkowski spacetime

$$X^{\mu} \rightarrow x_{A'A} = X^{\mu} (\sigma_{\mu})_{A'A} \tag{2.3}$$

with $\sigma_0 = 1$ and σ^i the Pauli matrices.

The inner product between two vectors reads

$$2X^{\mu} Y^{\nu} \eta_{\mu\nu} = \epsilon^{AB} \bar{\epsilon}^{A'B'} x_{A'A} y_{B'B}, \tag{2.4}$$

where $\epsilon = \mathrm{i}\sigma_2$ is the totally antisymmetric tensor.

From the above equation it follows that

$$\det x_{A'A} = X^{\mu} X_{\mu}. \tag{2.5}$$

For a null vector X^μ, the determinant of the corresponding matrix vanishes and therefore

$$x_{A'A} = \bar{c}_{A'} c_A, \tag{2.6}$$

in terms of a non-zero element of $\mathbf{C}^2$, which is called a *spinor*. Hence for each spinor c_A there corresponds one null vector

$$I^\mu = \bar{c}\sigma^\mu c, \tag{2.7}$$

where the indices are suppressed and summation is implied.

If a spinor c_A corresponds to a null vector I^μ, so does $\mathrm{e}^{\mathrm{i}\phi} c_A$. For this reason, the map from the space of non-zero spinors $\mathbf{C^2} - \{0\}$ to the space of null vectors on Minkowski spacetime, is many-to-one. Map (2.7) then defines a principal fibre bundle (the Hopf bundle), whose base space is the space V_+ of future-pointing null vectors (topologically $R \times S^2$) with positive energy ($I^0 > 0$),[2] total space is $\mathbf{C}^2 - \{0\}$ (topologically $R \times S^3$), fibre $U(1)$ and the projection map being defined by means of equation (2.7).

If I and J are two null vectors with corresponding spinors c and d their product is

$$2I_\mu J^\mu = |c_A \epsilon^{AB} d_B|^2. \tag{2.8}$$

In the following, we shall choose a reference cross-section of the Hopf bundle, by which a unique spinor ι represents the null vector I^μ. The most convenient choice is to consider spinors of the form $\binom{\mathrm{e}^\rho}{\mathrm{e}^\rho z}$, for any real ρ and complex number z.

The Hopf bundle is non-trivial, hence this cross-section is not global; it cannot be defined on the spinor $\binom{0}{1}$. But for all other spinors there exists an one-to-one map between future-directed null vectors and spinors, which reads explicitly:

$$I^\mu \to \iota = \begin{pmatrix} \sqrt{\frac{1}{2}(I^0 + I^3)} \\ \frac{I^1 + \mathrm{i}I^2}{\sqrt{2(I^0 + I^3)}} \end{pmatrix}. \tag{2.9}$$

We can, nonetheless, make the definition of ι unique by choosing $\iota = \binom{0}{1}$ for $I^\mu = (1, -1, 0, 0)$.

On $\mathbf{C}^2$ there exists the defining action of the $SL(2, \mathbf{C})$ group, i.e. of complex matrices with determinant one. For each $\alpha \in SL(2, \mathbf{C})$ one may define an element Λ of the Lorentz group as

$$\Lambda^{\mu\nu} = \tfrac{1}{2} \operatorname{Tr}(\alpha^\dagger \sigma^{(\mu} \alpha \sigma^{\nu)}). \tag{2.10}$$

The map is two-to-one since $\pm\alpha$ correspond to the same Lorentz matrix Λ.

A pair of spinors ι, j, such that $\iota^A \epsilon_{AB} j^B = 1$ defines an *orthonormal null tetrad* of vectors

$$I^\mu = \iota^* \sigma^\mu \iota \tag{2.11}$$

$$J^\mu = j^* \sigma^\mu j \tag{2.12}$$

$$m_1^\mu = \frac{1}{2}(\iota^* \sigma^\mu j + j^* \sigma^\mu \iota) \tag{2.13}$$

$$m_2^\mu = \frac{1}{2\mathrm{i}}(\iota^* \sigma^\mu j - j^* \sigma^\mu \iota), \tag{2.14}$$

which satisfy the equations

$$\eta^{\mu\nu} = \tfrac{1}{2}(I^\mu J^\nu + I^\nu J^\mu) - m_1^\mu m_1^\nu - m_2^\mu m_2^\nu. \tag{2.15}$$

$$m_1^\mu m_2^\nu - m_1^\nu m_2^\mu = \tfrac{1}{2} \epsilon^{\mu\nu\rho\sigma} I_\rho J_\sigma. \tag{2.16}$$

[2] An analogous fibre bundle may be defined for null vectors with negative energy.

2.3. Generalized coherent states

One may define generalized coherent states[3] using the representation of a group G by unitary operators $\hat{U}(g)$, $g \in G$ on a Hilbert space H. Selecting a reference vector $|0\rangle$ we may construct the vectors $\hat{U}(g)|0\rangle$. The usual choice for $|0\rangle$ is either the minimum energy state or a vector that is invariant under the maximal compact subgroup of G. We then define the equivalence relation on G as follows:

$$g \sim g' \text{ if there exists } \mathrm{e}^{\mathrm{i}\theta} \in U(1) \text{ such that } \hat{U}(g)|0\rangle = \mathrm{e}^{\mathrm{i}\theta}\hat{U}(g')|0\rangle.$$

Defining the manifold $\Gamma = G/\sim$, the map

$$[g] = z \in \Gamma \rightarrow \hat{U}(g)|0\rangle\langle 0|\hat{U}^{\dagger}(g) \tag{2.17}$$

defines a set of generalized coherent states $|z\rangle$, which possesses a resolution of the unity.

Through the generalized coherent states we may define a $U(1)$ connection on Γ

$$\mathrm{i}A = \langle z|dz\rangle, \tag{2.18}$$

which is familiar from the theory of geometric quantization [2, 12]. The closed 2-form $\Omega = \mathrm{d}A$ on Γ is in general degenerate, but if it is not it equips Γ with the structure of a symplectic manifold. In that case the Liouville form $\Omega \wedge \ldots \wedge \Omega$ defines an integration measure on Γ and suggests the existence of a resolution of the unity.

The generalized coherent states also allow the introduction of a Riemannian metric $\mathrm{d}s^2$ on Γ

$$\mathrm{d}s^2 = \langle \mathrm{d}z|\mathrm{d}z\rangle - |\langle z|\mathrm{d}z\rangle|^2. \tag{2.19}$$

The metric $\mathrm{d}s^2$ defines a notion of distance on Γ and incorporates the information about the uncertainty relation on phase space, namely the resolution in the determination of phase-space properties. In previous work [4], we proved that the condition $\delta s^2 \sim 1$ is equivalent to the Heisenberg uncertainty relations. The metric together with the connection allows the determination of the coherent state propagator $\langle z|\mathrm{e}^{-\mathrm{i}\hat{H}t}|z'\rangle$ by means of a path integral

$$\langle z''|\mathrm{e}^{-\mathrm{i}\hat{H}t}|z'\rangle = \lim_{\nu\to\infty} \int Dz(\cdot)\, \mathrm{e}^{\nu t}\, \mathrm{e}^{\mathrm{i}\int A - \mathrm{i}\int_0^t \mathrm{d}s H - \frac{1}{2\nu}\int_0^t \mathrm{d}s g_{ij}\dot{z}^i\dot{z}^j}, \tag{2.20}$$

where the integral is over all paths $z(\cdot)$ such that $z(0) = z'$ and $z(t) = z''$.

3. Generalized coherent states for massive particles

3.1. The representation of the Poincaré group

The unitary irreducible representations of the Poincaré group may be constructed by Wigner's procedure. We refer to the books of Simms [13] and Bogolubov *et al* [14] for a comprehensive treatment, upon which we base the constructions of the present paper.

The first step in Wigner's procedure involves the selection of a reference unit timelike vector and identification of its little group. We choose the vector $n^{\mu} = (1, 0, 0, 0)$. The corresponding little group is the group $SO(3)$ of spatial rotations. Any element Λ of the Lorentz group may be written as a product $\Lambda = \Lambda_I R$, where R is a rotation—element of the little group—and Λ_I is a boost taking n^{μ} to an arbitrary unit timelike vector I^{μ}

$$(\Lambda_I)^{\mu}{}_{\nu} n^{\nu} = I^{\mu}. \tag{3.1}$$

[3] In the present paper we consider as generalized coherent states any set of Hilbert space vectors labelled by points of a manifold, forming an overcomplete basis and possessing a resolution of the unity.

The boosts Λ_I read explicitly as

$$(\Lambda_I)^\mu{}_\nu = \delta^\mu{}_\nu + \frac{1}{I^0 - 1}(n^\mu - I^\mu)(n_\nu - I_\nu). \tag{3.2}$$

In the spinor representation n^μ corresponds to the *unit* 2×2 matrix, while Λ_I corresponds to the Hermitian matrix ω_I

$$\omega_I = \sqrt{\tilde{I}} = \frac{1}{\sqrt{2(1+I^0)}}(1+\tilde{I}), \tag{3.3}$$

where $\tilde{I} = I^\mu \sigma_\mu$. The fact that ω_I is a positive matrix and the existence of a polar decomposition for any matrix implies that an element of $SL(2, \mathbf{C})$ may be written as $\omega_I u$, where u is a unitary 2×2 matrix.

The unitary irreducible representations of the $SL(2, \mathbf{C})$ group are classified by means of the unitary irreducible representations of $SU(2)$, which is the universal cover of the little group $SO(3)$. It is well known that the representations of $SU(2)$ are characterized by an integer r, which labels the dimension of the representation's Hilbert space. We will denote by $D^{(r)}(g)$ the unitary $r \times r$ matrix representing the element $g \in SU(2)$.

To construct the representing Hilbert space we consider the space W_+ of unit time-like vectors ξ^μ with positive value of $\xi^0 = \sqrt{1+\xi^2}$, which is topologically homeomorphic to $\mathbf{R}^3$. W_+ may be equipped with the measure

$$\mathrm{d}\mu_M(\xi) = M^2\, \mathrm{d}^4\xi\, \delta(\xi^2 - 1) = M^2 \frac{\mathrm{d}^3\xi}{2\xi^0}, \tag{3.4}$$

which is labelled by the value M of the particle's rest mass. The introduction of this measure defines the Hilbert space $\mathcal{L}^2(W_+, \mathrm{d}\mu_M)$.

The Poincaré group is represented on Hilbert spaces $H_{M,r} = \mathcal{L}^2(W_+, \mathrm{d}\mu_M) \otimes \mathbf{C}^r$, which depend on the value of M and the integer r labelling a representation of $SU(2)$. The corresponding unitary operators $\hat{U}(\alpha, X)$ are defined as

$$[\hat{U}(\alpha, X)\Psi](\xi) = \mathrm{e}^{-\mathrm{i}M\xi\cdot X} D^{(r)}\big(\omega_\xi^{-1}\alpha\omega_{\alpha^{-1}\cdot\xi}\big)\Psi(\alpha^{-1}\cdot\xi), \tag{3.5}$$

where $\alpha \in SL(2, \mathbf{C})$, X^μ correspond to the Abelian group of spacetime translations, $\Psi(\xi) \in H_{M,n}$. The expression $\alpha \cdot \xi$ denotes the adjoint action $\alpha\tilde{\xi}\alpha^\dagger$ of α on the matrix $\tilde{\xi}_{A'A}$ corresponding to the vector ξ^μ.

3.2. The construction

We next select a reference vector to define the generalized coherent states. A vacuum state does not exist for free particle, and also no vectors are invariant under the maximal compact subgroup of the Poincaré group ($SO(3)$), unless the spin vanishes. Hence, there exist no natural candidates for a reference vector and our choice will be guided by calculational convenience. It should be noted that many of the results—such as the structure of the symplectic manifold parametrizing the generalized coherent states—do not depend on the explicit choice of the reference vector. However the Riemannian metric on the state space depends explicitly on that choice.

We choose a Gaussian vector $\psi_0 \in \mathcal{L}^2(W_+, \mathrm{d}\mu_M)$,

$$\psi_0(\xi) = \frac{1}{M(\pi\sigma^2)^{3/4}}(2n\cdot\xi)^{1/2}\, \mathrm{e}^{-\frac{1}{2\sigma^2}\xi^n\cdot\xi\cdot\xi}, \tag{3.6}$$

where ${}^n\xi_{\mu\nu} = -\eta_{\mu\nu} + n_\mu n_\nu$. This vector is centred around $\xi^i = 0$ with a width equal to σ.

We also choose a reference vector $|0\rangle_r$ on $\mathbf{C}^r$

$$|0\rangle_r = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}. \tag{3.7}$$

Then we may write a normalized reference vector on $H_{M,r}$,

$$\Psi_0(\xi) = \psi_0(\xi) \frac{D^{(r)}\big(\omega_\xi^{-1}\big)|0\rangle_r}{\sqrt{{}_r\langle 0|\tilde{\xi}^{-1}|0\rangle_r}}, \tag{3.8}$$

where we extended the use of the symbol $D^{(r)}$ to refer to the (non-unitary) representation of the $SL(2, \mathbf{C})$ associated with the r-dimensional representation of $SU(2)$.[4] The vector Ψ_0 is centred around the momentum value $\xi^i = 0$, and the spin pointing at the (0, 1, 0, 0) direction.

The action of $\hat{U}(\alpha, X)$ on Ψ_0 yields

$$\mathrm{e}^{-\mathrm{i}MX\cdot\xi}\psi_0(\alpha^{-1}\cdot\xi)\frac{D^{(r)}\big(\omega_\xi^{-1}\big)D^{(r)}(\alpha)|0\rangle_r}{\sqrt{{}_r\langle 0|D^{(r)}(\alpha)^\dagger\tilde{\xi}^{-1}D^{(r)}(\alpha)|0\rangle_r}}. \tag{3.9}$$

If we effect the polar decomposition of the $SL(2, \mathbf{C})$ matrix $\alpha = \omega_I u$, the $SU(2)$ matrix u will act on the reference vector on $\mathbf{C}^r|0\rangle_r$, leading to the generalized coherent states of the group $SU(2)$ $|\hat{\mathbf{m}}\rangle_r$

$$D^{(r)}(u)|0\rangle_r \to |\hat{\mathbf{m}}\rangle_r, \tag{3.10}$$

which are parametrized by a unit 3-vector $\hat{\mathbf{m}}$ [15]. If we denote by $\tilde{\mathbf{m}}$ the spinors corresponding to the 3-vector $\hat{\mathbf{m}}$, the inner product between the $SU(2)$ generalized coherent states reads

$${}_r\langle\hat{\mathbf{m}}_1|\hat{\mathbf{m}}_2\rangle_r = (\tilde{\mathbf{m}}_1^* \cdot \tilde{\mathbf{m}}_2)^r. \tag{3.11}$$

In terms of the $SU(2)$ coherent states, we define the following family of Hilbert space vectors:

$$\Psi_{I,\mathbf{m},X}(\xi) = \frac{1}{M(\pi\sigma^2)^{3/4}}(2I\cdot\xi)^{1/2}\,\mathrm{e}^{-\frac{1}{2\sigma^2}\xi\cdot I\xi\cdot\xi-\mathrm{i}MX\cdot\xi}\frac{D^{(r)}\big(\omega_\xi^{-1}\big)D^{(r)}(\omega_I)|\hat{\mathbf{m}}\rangle_r}{\sqrt{{}_r\langle\hat{\mathbf{m}}|D^{(r)}(\omega_I)^\dagger\tilde{\xi}^{-1}D^{(r)}(\omega_I)|\hat{\mathbf{m}}\rangle_r}}. \tag{3.12}$$

The unit timelike 4-vector I^μ is obtained by the action of the Lorentz transformation corresponding to α on the reference vector n^μ. It represents the particle's 4-momentum normalized to unity. The unit 3-vector $\hat{m}^i$ corresponds to the direction of the particle spin on a hypersurface normal to n^μ. It is more convenient to employ the unit, spacelike, 4-vector J^μ defined as

$$J = \Lambda_I\begin{pmatrix} 0 \\ \hat{\mathbf{m}} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{m}}\cdot\mathbf{I} \\ \big(\delta^{ij} - \frac{I^iI^j}{I^0-1}\big)\hat{m}^j \end{pmatrix}. \tag{3.13}$$

The 4-vector J_μ satisfies $I\cdot J = 0$ and is related to the Pauli–Lubanski vector by $W^\mu = M\frac{r}{2}J^\mu$.

[4] There exist two possible extensions of $SU(2)$ representations to the ones of $SL(2,\mathbf{C})$, depending on the embedding of $SU(2)$ in $SL(2,\mathbf{C})$ in the fundamental representation. If A is an $SU(2)$ matrix one may define the map $A \in SU(2) \to A \in SL(2,\mathbf{C})$, or the map $A \in SU(2) \to \epsilon\bar{A}\epsilon^{-1}$, where $\epsilon = \mathrm{i}\sigma_2$. The reference vectors do depend on that choice; however, the properties of the generalized coherent states are not affected. We shall employ the first alternative in the present paper.

The family of vectors above may be represented by a ket $|X, I, J\rangle_{M,r}$, which is parametrized by elements (X, I, J) of the nine-dimensional space $\Gamma_{M,r} = \mathbf{R}^7 \times S^2$. The action of the Poincaré group leaves this set of Hilbert space vectors invariant, in the sense that

$$\hat{U}(\Lambda, 0)|X, I, J\rangle = |\Lambda X, \Lambda I, \Lambda J\rangle \tag{3.14}$$

$$\hat{U}(0, Y)|X, I\rangle = |X + Y, I\rangle. \tag{3.15}$$

It should be emphasized that the spin degrees of freedom, encoded in the normalized Pauli–Lubanski vector J are continuous and hence $|X, I, J\rangle_{M,r}$ is labelled by the parameters of the classical state space, as appearing in the theory of Konstant–Souriau.

The space spanned by X, I, J is odd-dimensional and for this reason it is not expected to possess a resolution of the unity. The vectors $|X, I, J\rangle_{M,r}$ do not define therefore a family of generalized coherent states. One of the parameters in the set of vectors above plays the role of time and it has to be excised for a genuine family of generalized coherent states to be constructed. Classically, one defines the space of true degrees of freedom, by taking the quotient with respect to the action of the subgroup of time translations—the classical state space $\Gamma_{M,r}$ then consists of all classical solutions to the equations of motion, i.e. as the space of all orbits $(X, I, J)(s) = (X_0 + MI_0 s, I_0, J_0)$, with (X_0, I_0, J_0) a reference point. We may then define a set of generalized coherent states $|X(\cdot), I, J\rangle$, where $X(\cdot)$ is a path that solves the classical equations of motion. With this parametrization, the set of generalized coherent states transforms covariantly—similarly to equations (3.14), (3.15)—under the action of the Poincaré group and it is equipped with a resolution of the unity. To see this one may reduce the set of vectors $|X, I, J\rangle$, by taking a fixed value of the parameter $t = n \cdot X$, i.e. treating t as an external parameter and not as an argument of the generalized coherent states.

We then define the generalized coherent states at an instant of time, i.e. a spacelike 3-surface Σ, which is uniquely determined by the choices of n^μ and t. The generalized coherent states then depend on the spatial variables x^i and I^i, which are the projections of X and I on Σ together with the unit vector $\hat{m}^i$ of spin. These variables span the phase space of a single particle $T^*\Sigma \times S^2$, which is essentially the same with the covariant phase space spanned by the variables $X(\cdot), I, J$.

We denote the generalized coherent states on Σ as $|x, I, \hat{m}\rangle_\Sigma$. The Poincaré group behaves as follows: transformations that leave Σ invariant (spatial rotations and translations) preserve the generalized coherent states, while the ones that take Σ to another surface Σ' (namely boosts and time translations) also take the set of generalized coherent states into that associated with Σ'.

We may explicitly compute

$$\int \mathrm{d}^3 I\, \mathrm{d}^3 x\, \mathrm{d}^2 \hat{m} \langle \xi | x, I, \hat{m}\rangle_{\Sigma\,\Sigma}\langle x, I, \hat{m}|\xi'\rangle = \frac{1}{M^3} 2\kappa \omega_\xi \delta^3(\xi - \xi'), \tag{3.16}$$

a result that implies the existence of a resolution of the unity

$$\kappa \hat{1} = M^3 \int \mathrm{d}^3 I\, \mathrm{d}^3 x\, \mathrm{d}^2 \hat{m} |x, I, \hat{m}\rangle_{\Sigma\,\Sigma}\langle x, I, \hat{m}|. \tag{3.17}$$

Here κ equals the mean value of energy in the vector ψ_0

$$\kappa = \int \mathrm{d}\mu(\xi) \omega_\xi |\psi_0|^2(\xi). \tag{3.18}$$

Given a resolution of the unity, one may provide natural definitions of operators on $H_{M,r}$ in terms of functions on the classical phase space. Hence for any function $f : T^*\Sigma \times S^2 \to \mathbf{R}$, we may define the operator $\hat{F}_\Sigma$ as

$$\hat{F}_\Sigma = M^3 \int \frac{\mathrm{d}^3 I\, \mathrm{d}^3 x\, \mathrm{d}^2 \hat{m}}{\kappa} f(x, I, \hat{m}) |x, I, \hat{m}\rangle_{\Sigma\,\Sigma}\langle x, I, \hat{m}|. \tag{3.19}$$

We should note here that the operators $\hat{F}_\Sigma$ do not transform covariantly under the action of the Poincaré group. If a Poincaré transformation takes a 3-surface Σ to a 3-surface Σ', it *does not follow* that $\hat{F}_\Sigma$ is related to $\hat{F}_{\Sigma'}$ by means of the unitary operator corresponding to that Poincaré transformation. In particular, if Σ_t and $\Sigma_{t'}$ are two surfaces, corresponding to two different moments of time with respect to the same foliation, it does not follow that

$$\mathrm{e}^{\mathrm{i}\hat{H}(t'-t)}\hat{F}_{\Sigma_t}\,\mathrm{e}^{-\mathrm{i}\hat{H}(t'-t)} = \hat{F}_{\Sigma_{t'}}. \tag{3.20}$$

For example, we may consider the position operators $\hat{x}^i_\Sigma$, which represent length measurements only on the surface Σ. The Hamiltonian evolution yields

$$\mathrm{e}^{\mathrm{i}\hat{H}(t'-t)}\hat{x}_{\Sigma_t}\,\mathrm{e}^{-\mathrm{i}\hat{H}(t'-t)} = \hat{x}_{\Sigma_{t'}} + M\hat{I}_{\Sigma_{t'}}(t'-t). \tag{3.21}$$

It is often stated that the non-covariance of the position operator implies that particle position is not well defined in relativistic quantum mechanics. However, it needs to be noted that the index Σ does not refer to Heisenberg-time evolution, but is a kinematical parameter determining the reference frame that is involved in the specification of the corresponding measurement. In the consistent histories approach to quantum theory, the distinction between the kinematical and dynamical aspect of the change in physical parameters has a nice mathematical implementation [17], and there exists no conflict with covariance in position being represented by means of an one-parameter family of operators $\hat{x}^i_{\Sigma_t}$ [18].

We will next compare the method we followed here with that of [7], in which spin is represented by discrete variables. The starting point of [7] is the manifold Γ_0, which is obtained as a quotient of the Poincaré group G modulo $SU(2)\times T$, where T is the one-dimensional subgroup of time translations. Γ_0 is essentially the classical phase space of a massive, spinless relativistic particle (topologically $\mathbf{R}^6$). A fibre bundle $E(G,\Gamma_0,\pi)$ is then naturally defined with total space the Poincaré group, Γ_0 as base space and the projection π defined through the corresponding quotient. To construct a family of generalized coherent states on $H_{M,r}$ one chooses $2r+1$ linearly independent normalized vectors $|\eta^i\rangle$ on $H_{M,r}$ and a section σ of the bundle $E(G,\Gamma_0,\pi)$. The generalized coherent states are then defined as

$$|\xi,\mathrm{i}\rangle = \hat{U}(\sigma(\xi))|\eta^i\rangle, \tag{3.22}$$

where $\xi\in\Gamma$. These generalized coherent states possess a resolution of the unity. One may easily discern that the space spanned by $|\xi,i\rangle$ is identical with $2s+1$ copies of $\mathbf{R}^6$, namely it describes positions and momenta continuously and spin discretely.

The present method considers the action of the full Poincaré group on one reference vector of $H_{M,r}$. The bundle $E(G,\Gamma_0,\pi)$ is nowhere involved in this procedure either explicitly or implicitly and for this reason our results do not depend on the choice of a cross-section. The present method is the standard one for obtaining generalized coherent states associated with a group. We do not assume here an *a priori* distinction between momenta positions and spin degrees of freedom, and for this reason spin and momentum are non-trivially intertwined in the resulting generalized coherent states. It is well known that this is the case for spinning relativistic particles. For this reason it is very difficult to relate directly the present construction with that of [7], in which the spin degrees of freedom are fundamentally distinguished from those of momentum. The transformation properties under the Poincaré group are substantially different.

We should also remark that the coherent state parameter space in the present method is not a quotient of the Poincaré group by any subgroup (except for the trivial case $s=0$), but is defined by the equivalence relation of vectors that correspond to the same ray (see section 2.3). This parameter space can be identified with a coadjoint orbit of the Poincaré group, which is classically identified with the (unique) classical state space of massive spinning particles.

3.3. The coherent states' geometry

3.3.1. Connection and symplectic form. We now proceed to study the geometry of the parameter space for the generalized coherent states. First we evaluate the connection 1-form. For this purpose, it is more convenient to start with equation (3.9) and parametrize the $SL(2, \mathbf{C})$ matrix α as

$$\alpha = \begin{pmatrix} a & b \\ c & e \end{pmatrix}, \tag{3.23}$$

in terms of the complex numbers a, b, c, e, such that $ae - bc = 1$.

We then obtain

$$\mathrm{d}\Psi_{IJX}(\xi) = \left[\frac{\xi \cdot \mathrm{d}I}{2I \cdot \xi} - \frac{\xi \cdot I \xi \cdot \mathrm{d}I}{\sigma^2} - \mathrm{i}M\xi \cdot \mathrm{d}X\right]\Psi_{IJX}(\xi) + \psi_0(\xi)\left(\frac{D^{(r)}\big(\omega_\xi^{-1}\,\mathrm{d}\alpha\big)|0\rangle}{\sqrt{\langle 0|D^{(r)}\big(\alpha^\dagger\tilde{\xi}^{-1}\alpha\big)|0\rangle}}\right.$$
$$\left. - \frac{1}{2}\frac{\langle 0|D^{(r)}(\alpha^\dagger\tilde{\xi}^{-1}\,\mathrm{d}\alpha)|0\rangle + \langle 0|D^{(r)}(\mathrm{d}\alpha^\dagger\tilde{\xi}^{-1}\alpha)|0\rangle}{(\langle 0|D^{(r)}(\alpha^\dagger\tilde{\xi}^{-1}\alpha)|0\rangle)^{3/2}} D^{(r)}\big(\omega_\xi^{-1}\,\mathrm{d}\alpha\big)|0\rangle\right). \tag{3.24}$$

In order to compute the expression $\langle X, I, J|d|X, I, J\rangle$, which involves integration over $\mathrm{d}\mu_M(\xi)$ we perform the change of variables $\tilde{\xi} \to \alpha^{-1} \cdot \tilde{\xi}$. We also use the following relation,

$${}_r\langle 0|D^{(r)}(\beta)|0\rangle_r = ({}_2\langle 0|\beta|0\rangle_2)^r, \tag{3.25}$$

which enables us to compute all inner products in the fundamental representation of $SU(2)$ on $\mathbf{C}^2$.

The first term in $\langle X, I, J|d|X, I, J\rangle$ reads

$$\mathrm{i}M\kappa I^\mu\,\mathrm{d}X_\mu, \tag{3.26}$$

while the second

$$\frac{r}{2}[(e\,\mathrm{d}a - b\,\mathrm{d}c) - (e^*\,\mathrm{d}a - b^*\,\mathrm{d}c^*)], \tag{3.27}$$

which may be written as

$$\frac{l}{2}[\lambda_A\epsilon^{AB}\,\mathrm{d}\mu_B - \lambda^*_{A'}\epsilon^{A'B'}\,\mathrm{d}\mu^*_{B'}], \tag{3.28}$$

in terms of the two spinors

$$\mu = \begin{pmatrix} a & b \\ c & e \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \tag{3.29}$$

$$\lambda = \begin{pmatrix} a & b \\ c & e \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ e \end{pmatrix}. \tag{3.30}$$

The spinor μ is obtained by a Lorentz transformation of the spinor $\binom{1}{0}$, which corresponds to the null vector $(1, 1, 0, 0)$. Hence, λ corresponds to the null vector $I + J$. Similarly, the spinor μ is obtained by a Lorentz transformation of the spinor $\binom{0}{1}$, which corresponds to the null vector $(1, -1, 0, 0)$. Hence, λ corresponds to the null vector $I - J$. The two spinors satisfy $\lambda_A\epsilon^{AB}\mu_B = 1$. They, therefore, define a null tetrad.

The final result is

$$A = -\kappa M I^\mu\,\mathrm{d}X_\mu - \frac{\mathrm{i}r}{2}[\lambda_A\epsilon^{AB}\,\mathrm{d}\mu_B - \lambda^*_{A'}\epsilon^{A'B'}\,\mathrm{d}\mu^*_{B'}]. \tag{3.31}$$

We may absorb κ in a redefinition of the mass M as $M' = \kappa M$, or in a redefinition of the spacetime coordinates $Y^\mu = \kappa X^\mu$. We shall prefer here the latter alternative.

Under the gauge transformation $\mu \to \mathrm{e}^{\mathrm{i}\theta}\mu$, $\lambda \to \mathrm{e}^{-\mathrm{i}\theta}\lambda$, the connection form transforms as $A \to A + r\,\mathrm{d}\theta$, while the 2-form

$$\Omega = M\,\mathrm{d}I^\mu \wedge \mathrm{d}Y_\mu - \mathrm{i}\frac{r}{2}[\mathrm{d}\lambda_A \wedge \epsilon^{AB}\,\mathrm{d}\mu_B - \mathrm{d}\lambda^*_{A'} \wedge \epsilon^{A'B'}\,\mathrm{d}\mu^*_{B'}], \tag{3.32}$$

remains invariant. Ω may also be written in terms of the vectors I and J as [2]

$$\Omega = M\,\mathrm{d}I^\mu \wedge \mathrm{d}Y_\mu - \frac{r}{4}\epsilon_{\mu\nu\rho\sigma} I^\mu J^\nu (\mathrm{d}I^\rho \wedge \mathrm{d}I^\sigma - \mathrm{d}J^\rho \wedge \mathrm{d}J^\sigma). \tag{3.33}$$

The 2-form Ω is degenerate: the degenerate direction corresponds to the vector field $I^\mu \frac{\partial}{\partial Y^\mu}$.

Through the generalized coherent states, we have recovered the standard form of the state space and symplectic structure of spinning relativistic particles with non-zero mass.

3.3.2. The metric. The calculation of the Riemannian metric on $\Gamma_{M,r}$ is straightforward but tedious. The end result is the following:

$$\mathrm{d}s^2 = \mathrm{d}s_0^2 + \frac{\mathrm{i}r}{4}\kappa M[(\lambda\,\tilde{\mathrm{d}}X\mu^*)(\mu\epsilon\,\mathrm{d}\mu) - (\mu\,\tilde{\mathrm{d}}X\lambda^*)(\mu^*\epsilon\,\mathrm{d}\mu^*)] + \frac{r^2}{4}(1-v)|\mu\epsilon\,\mathrm{d}\mu|^2. \tag{3.34}$$

Here v denotes the constant

$$v = 2\int \mathrm{d}\mu_M(\xi)|\psi_0|^2(\xi)\frac{\xi^3}{\xi^0+\xi^3} \tag{3.35}$$

and $\mathrm{d}s_0^2$ the corresponding metric for the spinless relativistic particles

$$\mathrm{d}s_0^2 = -\frac{\omega}{3\sigma^2}\eta_{\mu\nu}\,\mathrm{d}I^\mu\,\mathrm{d}I^\nu + K_{\mu\nu}\,\mathrm{d}X^\mu\,\mathrm{d}X^\nu. \tag{3.36}$$

The first term is the Riemannian metric on W_+ inherited from the Lorentzian metric on Minkowski spacetime times a constant. The parameter ω equals

$$\omega = \frac{1}{(\pi\sigma^2)^{1/2}}\int_0^\infty \frac{\mathrm{d}\xi}{1+\xi^2}\,\mathrm{e}^{-\xi^2/\sigma^2}. \tag{3.37}$$

The second term involves the tensor

$$K^{\mu\nu} = \langle X, I|\hat{P}^\mu P^\nu|X, I\rangle - \langle X, I|\hat{P}^\mu|X, I\rangle\langle X, I|\hat{P}^\nu|X, I\rangle, \tag{3.38}$$

which is the correlation tensor for the 4-momentum on a coherent state. Explicitly,

$$K_{\mu\nu} = M^2\left[\left(1 + \tfrac{2}{3}\sigma^2 - \kappa^2\right)I_\mu I_\nu - \tfrac{1}{6}\sigma^2\eta_{\mu\nu}\right]. \tag{3.39}$$

One may choose $\sigma^2 \ll 1$, in which case the reference vector approaches weakly a delta function on momentum space. In that case, the parameters κ, ω, v behave as

$$\kappa = 1 + \tfrac{1}{4}\sigma^2 - \tfrac{1}{16}\sigma^4 + O(\sigma^6), \tag{3.40}$$

$$\omega = \tfrac{1}{2} + O(\sigma^2), \tag{3.41}$$

$$v = O(\sigma^2). \tag{3.42}$$

This implies that the dominant contribution to the phase space metric for small σ^2 is

$$\begin{aligned}\mathrm{d}s^2 = -\frac{1}{6\sigma^2}\eta_{\mu\nu}\,\mathrm{d}I^\mu\,\mathrm{d}I^\nu + M^2\frac{\sigma^2}{6}(I_\mu I_\nu - \eta_{\mu\nu})\delta X^\mu\delta X^\nu + \frac{\mathrm{i}r}{4}\kappa M[(\lambda\,\mathrm{d}\tilde{X}\mu^*)(\mu\epsilon\,\mathrm{d}\mu) \\ - (\mu\,\tilde{\mathrm{d}}X\lambda^*)(\mu^*\epsilon\,\mathrm{d}\mu^*)] + \frac{r^2}{4}(1-v)|\mu\epsilon\,\mathrm{d}\mu|^2.\end{aligned} \tag{3.43}$$

Note that this metric has a degenerate direction, which coincides with that of the symplectic form (3.32).

In the particle's rest frame $I^i = 0$ and for $t = 0$, the spin-dependent terms in the metric read

$$\frac{r}{2} M\kappa \mathbf{m}\cdot(\mathrm{d}\mathbf{m}\times\mathrm{d}\mathbf{x}) + \frac{r^2}{4}\,\mathrm{d}\mathbf{m}\cdot\mathrm{d}\mathbf{m}. \tag{3.44}$$

The leading terms in the metric are quite important, as they are less dependent on the details of the chosen reference vector. For reasons of continuity, a small change in the reference vector (with respect to the Hilbert space norm) will have a smaller effect in the dominant terms. For this reason, the metric (3.43) is the most suitable candidate for the path-integral calculation of the coherent state overlap functional (2.20), which cannot be analytically computed with our Gaussian wavefunctions.

It is well known that knowledge of the overlap functional enables one to fully reconstruct the information about the Hilbert space and the coherent construction. Since we are using the metric (3.43) and not the full metric (3.43) of the generalized coherent states, the reference vector corresponding to that construction will be different from that we employed here. Still, the geometric structure of the generalized coherent states will remain the same.

4. Generalized coherent states for massless particles

4.1. The representation of the Poincaré group

The unitary irreducible representations of the Poincaré group for zero mass are very different from the massive ones; they may not be obtained as the $M \to 0$ limit of the massive representations. For this reason the structures of the corresponding generalized coherent states are quite different. Nonetheless, they may be constructed using the same method.

We follow again Wigner's procedure for the construction of the group's representation. For that purpose, we select a reference null vector and identify its little group. It is convenient to work in the spinor representation and take $\binom{1}{0}$ as a reference spinor. The corresponding little group consists of all matrices $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL(2,\mathbf{C})$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathrm{e}^{\mathrm{i}\phi}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.1}$$

for some phase $\mathrm{e}^{\mathrm{i}\phi}$. This is satisfied by all matrices of the form

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & \mathrm{e}^{-\mathrm{i}\theta} z \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix}. \tag{4.2}$$

Each unitary representation of the little group defines uniquely a unitary representation of the full Poincaré group. The unitary representations of this little group that are relevant to the description of massless particles are one-dimensional and correspond to the multiplication by a phase

$$\alpha = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & \mathrm{e}^{-\mathrm{i}\theta} z \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix} \to D_r(\alpha)\,\mathrm{e}^{-\mathrm{i}r\theta}, \tag{4.3}$$

where r is an integer that corresponds to the discrete values of spin. The representations with opposite values of r correspond to particles with the same spin but opposite helicity.

Any element of $SL(2,\mathbf{C})$ may be written as a product of a matrix of the form (4.3) with a matrix of the form

$$\begin{pmatrix} \mathrm{e}^{\rho} & 0 \\ \mathrm{e}^{\rho} z & \mathrm{e}^{-\rho} \end{pmatrix}. \tag{4.4}$$

For each null vector ξ^μ we denote as ω_ξ the unique matrix of type (4.4) that takes the reference spinor $\binom{1}{0}$ to the canonical spinor $\tilde{\xi}$ associated with ξ.[5] In effect if $\tilde{\xi} = e^\rho \binom{1}{z}$ then $\omega_\xi = \begin{pmatrix} e^\rho & 0 \\ e^\rho z & e^{-\rho} \end{pmatrix}$.

The massless representations are constructed on the Hilbert space $H_0 = L^2(V_+, d\mu(\xi))$ of complex-valued, square-integrable functions over the space V_+ of future-directed null vectors. The measure $d\mu(\xi)$ is the unique Poincaré invariant

$$d\mu(\xi) = \frac{d^3\xi}{2\xi}, \tag{4.5}$$

where $\xi = \sqrt{\xi \cdot \xi}$.

The representations are characterized by the integer r of spin

$$[\hat{U}[\Lambda, X]\Psi](\xi) = e^{iX\cdot\xi} D_r\left[\omega_\xi^{-1}\alpha(\Lambda)\omega_{\Lambda^{-1}\xi}\right]\Psi(\Lambda^{-1}\xi), \tag{4.6}$$

where $\alpha(\Lambda)$ is a $SL(2, \mathbf{C})$ matrix corresponding to the Lorentz matrix Λ.

4.2. The construction

We select a reference vector sharply concentrated around a specific element of V_+, conveniently chosen as $\xi^\mu = (1, 0, 0, 1)$. We thus need to identify smeared delta functions on the space V_+.

Unlike the massive case, V_+ has the topology $\mathbf{R} \times S^2$, because the null vector $(0, 0, 0, 0)$ is excluded. This implies that a (smeared) delta function on V_+ factorizes into a product of a delta function on $\mathbf{R}$ with a delta-function on S^2. However, the identification of the component of ξ^μ acting as coordinate on $\mathbf{R}$ and of the components acting as coordinates on S^2 is not Lorentz invariant. It depends on the choice of a reference timelike vector. Choosing $n_R^\mu = (1, 0, 0, 0)$, the coordinate $\xi = n_R^\mu \xi_\mu$ takes values in $(0, \infty)$. Hence the coordinate $\lambda = \log \xi^0$ runs across the full real line.

The sphere S^2 is essentially the celestial sphere corresponding to the timelike direction n_R. The reference null vector $(1, 0, 01)$ specifies a direction on this sphere corresponding to the spacelike unit vector $m_R^\mu = (0, 0, 0, 1)$. The smeared delta function should be a function of only the distance of the argument ξ^μ from the reference vector m_R^μ. It should be, therefore, a function of $m_R^\mu \xi_\mu = \xi \cos\theta$, where θ is the angle between the 3-vectors ξ and $\mathbf{m}_R$.

If we use as coordinates λ, $x = \cos\theta$ and ϕ (an azimuthal angle on the sphere running from 0 to 2π), the invariant measure becomes

$$d\mu(\xi) = e^{2\lambda}\, d\lambda\, dx\, d\phi. \tag{4.7}$$

It is convenient to employ a Gaussian as a smeared delta function for the variable λ

$$f(\lambda) = \frac{1}{\sqrt{\pi\sigma^2}} e^{-\frac{\lambda^2}{\sigma^2} - 2\lambda}. \tag{4.8}$$

For the sphere S^2 recall that the delta function with respect to the north pole is given by

$$\delta(x) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(x), \tag{4.9}$$

where $x = \cos\theta$ and P_l the standard (unnormalized) Legendre polynomials.

A convenient choice for a smeared delta function is to truncate the series at some value $l = N$. So the smeared delta function is

$$g(x) = \sum_{l=0}^{N} \frac{2l+1}{4\pi} P_l(x). \tag{4.10}$$

[5] In this section we denote as $\tilde{\xi}$ a spinor, while in the previous one it denoted the 2×2 matrix corresponding to ξ^μ.

The benefit from this choice of smearing function is that for any polynomial f of x of degree less than or equal to N, we have

$$2\pi \int_{-1}^{1} \mathrm{d}x\, g(x) f(x) = f(1). \tag{4.11}$$

With the previous choices of smeared delta functions we may write a reference vector on the Hilbert space H_0,

$$\Psi_0(\xi) = \sqrt{f}(\log n_R \cdot \xi)\sqrt{g}\left(\frac{m_R \cdot \xi}{n_R \cdot \xi}\right). \tag{4.12}$$

When the unitary operator $U[\alpha, X]$ acts on Ψ_0, the argument of Ψ_0 goes from $\tilde{\xi}$ to $\alpha^{-1}\tilde{\xi}$. Since Ψ_0 is a function of $n_R \cdot \xi$ and $m_R \cdot \xi$, this transformation renders Ψ_0 into a function of $(\Lambda(\alpha)n_R) \cdot \xi$ and $(\Lambda(\alpha)m_R) \cdot \xi$, where $\Lambda(\alpha)$ is the element of the Lorentz group corresponding to the $SL(2, \mathbf{C})$ matrix α. The generalized coherent states depend on $n = \Lambda n_R$ and $m = \Lambda m_R$, which are unit timelike and spacelike vectors respectively that satisfy $n \cdot m = 0$. It is more convenient to employ a pair of null vectors $I^\mu = n^\mu + m^\mu$, $J^\mu = n^\mu - m^\mu$, which satisfy $I_\mu J^\mu = 2$.

The non-trivial part of the construction is the one referring to the representation D_r of the little group. If we write ξ^μ in terms of its representative spinor $\mathrm{e}^\rho \binom{1}{z}$, and consider a general $SL(2, \mathbf{C})$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we get

$$D_r\left[\omega_\xi^{-1} \alpha \omega_{\Lambda^{-1}\xi}\right] = \left(\frac{d - bz}{|d - bz|}\right)^r. \tag{4.13}$$

The action of the $SL(2, \mathbf{C})$ matrix on $\binom{0}{1}$ gives $\binom{b}{d}$. But $\binom{0}{1}$ corresponds to $n_R - m_R^\mu = (1, 0, 0, -1)$ and hence $\binom{b}{d}$ corresponds to J^μ. Thus it can be written as $j\,\mathrm{e}^{\mathrm{i}\chi}$ for some phase χ. Taking this into account we see that

$$D_r\left[\omega_\xi^{-1} \alpha \omega_{\alpha^{-1}\xi}\right] = \left(\frac{\tilde{\xi}_A \epsilon^{AB} j^B\, \mathrm{e}^{\mathrm{i}\chi}}{|\tilde{\xi}_A \epsilon^{AB} j^B|}\right)^r. \tag{4.14}$$

However, the fact that $ad - bc = 1$ implies that χ must be absorbed in a redefinition of j such that

$$\iota^A \epsilon_{AB} j^B = 1, \tag{4.15}$$

so that the spinors ι and j define a null tetrad. One should note that—unlike the massive particles case—the vector J^μ is not here the normalized Pauli–Lubanski vector, since the latter is a multiple of I^μ in the massless case.

Eventually, using (4.14) we arrive at an expression for a set of vectors $|X, I, J\rangle_r$, from which we shall construct the generalized coherent states corresponding to the massless representations of the Poincaré group

$$\langle \xi | X, I, J\rangle_r = \Psi_{X,I,J}^{(r)} = \left(\frac{\tilde{\xi}_A \epsilon^{AB} j_B}{|\tilde{\xi}_A \epsilon^{AB} j_B|}\right)^r \mathrm{e}^{-\mathrm{i}\xi \cdot X} \sqrt{f}\left(\log \frac{I + J}{2} \cdot \xi\right)\sqrt{g}\left(\frac{-(I - J) \cdot \xi}{(I + J) \cdot \xi}\right). \tag{4.16}$$

The parameters X, I, J of these vectors span a nine-dimensional manifold, which we will call $\Gamma_{0,r}$. This is not, however, the phase space of a classical system. We have to take into account the fact that two different sets of parameters correspond to the same Hilbert space ray, i.e. that there might be a pair (X, I, J) and (X', I', J') such that

$$\langle X, I, J | X', I', J'\rangle = \mathrm{e}^{\mathrm{i}\phi}. \tag{4.17}$$

Writing $X' = X + \mathrm{d}X$, $I' = I + \mathrm{d}I$, $J' = J + \mathrm{d}J$, the above equation reads

$$\langle X, I, J|\mathrm{d}|X, I, J\rangle = \mathrm{i}\mathrm{d}\phi(X, I, J), \tag{4.18}$$

or in terms of the $U(1)$ connection A of (2.18)

$$A - \mathrm{d}\phi = 0. \tag{4.19}$$

One has, therefore, to excise all submanifolds of M in which the 1-form A becomes closed, or in other words remove all the degenerate directions of the symplectic form $\Omega = \mathrm{d}A$.

To compute A we first write $\mathrm{d}\Psi_{X,I,J}$

$$\begin{aligned}\mathrm{d}\Psi_{X,I,J} = &-\mathrm{i}\xi \cdot \mathrm{d}X\Psi_{X,I,J}(\xi) + \frac{f'}{2f}\left(\log\frac{1}{2}\xi\cdot(I+J)\right)\frac{\xi\cdot(\mathrm{d}I+\mathrm{d}J)}{\xi\cdot(I+J)}\Psi_{X,I,J}(\xi)\\ &-\frac{g'}{g}\left(\frac{(I-J)\cdot\xi}{(I+J)\cdot\xi}\right)\frac{(\xi\cdot J)(\xi\cdot\mathrm{d}I)-(\xi\cdot I)(\xi\cdot\mathrm{d}J)}{[(I+J)\cdot\xi]^2}\Psi_{X,I,J}(\xi)\\ &+\frac{r}{2}\frac{(\tilde{\xi}\epsilon\tilde{J})^*(\tilde{\xi}\epsilon\,\mathrm{d}j)-(\tilde{\xi}\epsilon j)(\tilde{\xi}\epsilon\,\mathrm{d}j)^*}{|\tilde{\xi}\epsilon\,\mathrm{d}j|^2}\Psi_{X,I,J}(\xi).\end{aligned} \tag{4.20}$$

It is convenient to change variables to $\xi' = \Lambda^{-1}\xi$, in order to compute the integral $\int \mathrm{d}\mu(\xi)\Psi^*_{X,I,J}(\xi)\,\mathrm{d}\Psi_{X,I,J}(\xi)$. The reference null directions become $I^\mu_R = (1, 0, 0, 1)$ and $J^\mu_R = (1, 0, 0, -1)$. In terms of these directions we can parametrize ξ' as

$$\tilde{\xi}' = \mathrm{e}^{\lambda}\begin{pmatrix}\sqrt{\frac{1+x}{2}}\\ \sqrt{\frac{1-x}{2}}\,\mathrm{e}^{\mathrm{i}\phi}\end{pmatrix}, \tag{4.21}$$

where $x = \cos\theta$ refers to the angle between ξ^i and $m^i_R = (0, 0, 1)$.

The evaluation of the integral is now straightforward. The first line of (4.20) gives a term $-\mathrm{i}\mathrm{e}^{\sigma^2/4}I^\mu\,\mathrm{d}X_\mu$. We can absorb the factor $\mathrm{e}^{\sigma^2/4}$ into a redefinition of X^μ, i.e. write $Y^\mu = \mathrm{e}^{\sigma^2/4}X^\mu$ so that the first term reads $-\mathrm{i}I^\mu\,\mathrm{d}Y_\mu$. The contributions of the second and third terms cancel each other, while the final term contributes $\frac{r}{2}(\iota\epsilon\,\mathrm{d}j - \iota^*\epsilon\,\mathrm{d}j^*)$. So the expression for the connection reads

$$A = -I^\mu\,\mathrm{d}Y_\mu - \frac{\mathrm{i}r}{2}(\iota\epsilon\,\mathrm{d}j - \iota^*\epsilon\,\mathrm{d}j^*). \tag{4.22}$$

If we define the spinor

$$\omega_A = j_A + \frac{2\mathrm{i}}{r}y_{A'A}\iota^{*A'}, \tag{4.23}$$

we obtain (up to a closed form)

$$A = \frac{\mathrm{i}r}{2}(\iota^A\epsilon_{AB}\,\mathrm{d}\omega^B - \iota^{*A'}\epsilon_{A'B'}\,\mathrm{d}\omega^{*B'}), \tag{4.24}$$

giving the symplectic form

$$\Omega = \frac{\mathrm{i}r}{2}(\mathrm{d}\iota^A \wedge \epsilon_{AB}\,\mathrm{d}\omega^B - \mathrm{d}\iota^{*A'} \wedge \epsilon_{A'B'}\,\mathrm{d}\bar{\omega}^{A'}). \tag{4.25}$$

If we consider the spinor ω^A as a function of Y—through equation (4.23)—then it satisfies the *twistor equation* (see for instance [16])

$$\nabla^{(A}_{A'}\omega^{B)}(Y) = 0, \tag{4.26}$$

where $\nabla_{A'A} = \sigma^\mu_{A'A}\partial_\mu$. Note that ι initially refers to the canonical expression (2.9) for the spinor corresponding to the null vector I^μ. Had it been unrestricted, the pair ι^A, ω_A would define an element of the twistor space $\mathbf{T}$, namely the space of solutions to equation (4.26).

However, we may allow variations of the phase of ι. In particular, under the transformation

$$\omega_A \to \omega_A \, \mathrm{e}^{\mathrm{i}\theta} \qquad \iota^A \to \iota^A \, \mathrm{e}^{-\mathrm{i}\theta} \tag{4.27}$$

the connection transforms

$$A \to A - r \, \mathrm{d}\theta, \tag{4.28}$$

which implies that the angle θ corresponds to a degenerate direction of the symplectic 2-form. Hence the generalized coherent states' parameter space Γ consists of equivalence classes of pairs (ι^A, ω_A) under transformation (4.27), which satisfy

$$\tfrac{1}{2}(\iota^A \epsilon_{AB} \omega^B + \iota^{*A'} \epsilon_{A'B'} \omega^{*B'}) = 1. \tag{4.29}$$

Equation (4.29) is due to definition (4.23). In particular, this equation implies that ι cannot vanish, in accordance with the fact that I^μ may not take the value (0, 0, 0, 0).

If we perform the transformation

$$\omega^A \to \zeta^A = \omega^A + \frac{2\mathrm{i}}{r} u j^A, \tag{4.30}$$

where $u = I^\mu Y_\mu$, we see that the spinor ζ^A satisfies $\iota^A \epsilon_{AB} \zeta^B = 1$. Hence the pair (ι^A, ζ^A) defines an orthonormal null tetrad. Moreover, ζ^A transforms under (4.27) as

$$\zeta^A \to \zeta^A \, \mathrm{e}^{\mathrm{i}\theta}, \tag{4.31}$$

a fact that implies that symmetry (4.27) of the symplectic form corresponds to a rotation of the spacelike vectors m_1 and m_2 of the null tetrad—see equations (2.13), (2.14). These vectors are not variables on the physical state space. Consequently, the space Γ may be parametrized by the null vectors I^μ, ζ^μ (with $I \cdot \zeta = 2$) corresponding to the spinors ι^A, ζ^A, together with the parameter u. Since I^μ cannot vanish, the topology of the resulting space is $\mathbf{R}^4 \times S^2$. Remarkably, the set of generalized coherent states $|\iota, \zeta, u\rangle$ is parametrized by the even-dimensional symplectic manifold Γ. This symplectic manifold does not depend on a choice of Lorentzian foliation. For this reason, the generalized coherent states transform covariantly (up to a phase) under the action of the Poincaré group:

$$\hat{U}(\alpha, 0) \, \mathrm{e}^{\mathrm{i}\chi} |\iota, \zeta, u\rangle \to = |\alpha\iota, \alpha\zeta, u\rangle \tag{4.32}$$

$$\hat{U}(1, C)|\iota, \zeta, u\rangle = \mathrm{e}^{\mathrm{i}\chi} |\iota, C \cdot \zeta, u + I \cdot C\rangle, \tag{4.33}$$

where $C \cdot \zeta$ denotes the nonlinear action of spacetime translations on ζ, by virtue of equations (4.23) and (4.30). The phase χ depends on our phase convention about the generalized coherent states. Clearly, the projection operators $|\iota, \zeta, u\rangle\langle\iota, \zeta, u|$ transform in a fully covariant manner under the Poincaré group.

4.2.1. The phase-space metric. The determination of the phase-space metric involves extensive calculations. We here present the final result

$$\begin{aligned} \mathrm{d}s^2 = (\mathrm{e}^{\sigma^2} - \mathrm{e}^{\sigma^2/2})(I \cdot \mathrm{d}X)^2 + \frac{1}{4}\left(1 + \frac{1}{2\sigma^2} + 3c_1\right)(I \cdot \mathrm{d}J)^2 - \frac{1}{2}\left(\frac{1}{4}c_2 + 1\right) \mathrm{d}I \cdot \mathrm{d}I \\ - \frac{1}{8} c_3 \, \mathrm{d}J \cdot \mathrm{d}J + \left(\frac{c_1}{4} - 1\right) \mathrm{d}I \cdot \mathrm{d}J + \frac{r^2}{2} F |j\epsilon \, \mathrm{d}j|^2, \end{aligned} \tag{4.34}$$

in terms of the coefficients

$$c_1 = 2\pi \int_{-1}^{1} \mathrm{d}x \, \frac{g'^2}{4g}(x)(1-x^2)^2 \tag{4.35}$$

$$c_2 = 2\pi \int_{-1}^{1} \mathrm{d}x \, \frac{g'^2}{4g}(x)(1-x^2)(1+x)^2 \tag{4.36}$$

$$c_3 = 2\pi \int_{-1}^{1} \mathrm{d}x \, \frac{g'^2}{4g}(x)(1-x^2)(1-x)^2 \tag{4.37}$$

$$F = 2\pi \int_{-1}^{1} \mathrm{d}x \, g(x) \frac{1-x}{1+x}. \tag{4.38}$$

As the smearing parameters $\sigma^2 \to 0$ and $N \to 0$ the smearing function approaches weakly a delta function on momentum space. In that case the metric simplifies. However, the smeared delta function (4.10), which has been very convenient in our calculations, is of limited use in the explicit computation of the coefficients (4.35)–(4.38). For this task we will employ a different smearing function on S^2. This change does not affect the behaviour of the dominant terms, except for the fact that they are written in terms of a different smearing parameter. We, therefore, employ in equations (4.35)–(4.38) the function

$$g(x) = \frac{1}{2\pi} C \frac{1+x}{(1-x)^2 + \epsilon^2}. \tag{4.39}$$

The coefficient is obtained from the normalization condition $\pi \int_{-1}^{1} \mathrm{d}x \, g(x) = 1$. Explicitly,

$$C = \left[\frac{2}{\epsilon} \tan^{-1}\left(\frac{2}{\epsilon}\right) + \log \frac{\epsilon}{2} \right]^{-1}. \tag{4.40}$$

We may then evaluate the coefficients (4.35)–(4.38)

$$c_1 = 4 + O(\epsilon) \tag{4.41}$$

$$c_2 = \frac{8}{\epsilon} + O(\epsilon^0) \tag{4.42}$$

$$c_3 = O(\epsilon) \tag{4.43}$$

$$F = \frac{\epsilon}{\pi} \log \frac{2}{\epsilon} + O(\epsilon^2). \tag{4.44}$$

Inspection of (4.34) shows that with the choice of $\epsilon = 8\sigma^2$ the leading behaviour of the metric takes a rather simple form

$$\mathrm{d}s^2 = \frac{\sigma^2}{2}(I \cdot \mathrm{d}X)^2 + \frac{1}{8\sigma^2}(J_\mu J_\nu - \eta_{\mu\nu}) \, \mathrm{d}I^\mu \, \mathrm{d}I^\nu + r^2 \frac{8\sigma^2}{\pi} \log \frac{1}{2\sigma} |j\epsilon \, \mathrm{d}j|^2. \tag{4.45}$$

5. Conclusions

We constructed the generalized coherent states corresponding to the physical unitary irreducible representations of the Poincaré group. The space of parameters for these states corresponds to the classical symplectic manifold that describes spinning relativistic particles. The description of these state spaces in terms of generalized coherent states is perhaps more accessible (if less elegant) to the particle physicist, because the standard classical derivation involves rather advanced techniques of symplectic geometry.

There are some differences and additions in our work, as compared to the results that have appeared in the bibliography. We briefly summarize them here.

- Our generalized coherent states are obtained in a straightforward manner from the group representations of the Poincaré group. The same procedure is followed, therefore, for both the massless and massive representations of the Poincaré group.
- The parameter space of the resulting generalized coherent states is identified with the classical state space of spinning relativistic particles, which correspond to the coadjoint orbits of the Poincaré group. This procedure highlights the distinction between massive and massless particles.
- Our choice for the reference vector allows us to perform explicit calculations, such as the Riemannian matric on state space, which is an essential ingredient of the coherent-state path integral.

Our results imply that one may write a phase-space representation of quantum theory for spinning particles and for the fields constructed from such particles. Geometric objects, such as the $U(1)$ connection and the Riemannian metric on phase space will play an important role in that description. It will be of great technical and conceptual interest [19] to explore the properties of quantum field theory in that particle representation. The present paper provides a stepping stone in that direction.

Acknowledgment

The research was partly supported by a Marie Curie Fellowship of the European Union.

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